

Reflector imaging from ambient noise signals

The goal is to show numerically that the cross correlations of signals emitted by noise sources and recorded by a receiver array can be processed to localize reflectors.

1) Geometric set-up.

We consider the solution u of the wave equation in a three-dimensional medium:

$$\frac{1}{c(\mathbf{x})^2} \frac{\partial^2 u}{\partial t^2} - \Delta_{\mathbf{x}} u = n(t, \mathbf{x}). \quad (1)$$

The term $n(t, \mathbf{x})$ models a random field of noise sources. It is a zero-mean stationary (in time) random process with autocorrelation function

$$\langle n(t_1, \mathbf{y}_1) n(t_2, \mathbf{y}_2) \rangle = F(t_2 - t_1) \delta(\mathbf{y}_1 - \mathbf{y}_2) K(\mathbf{y}_1). \quad (2)$$

Here $\langle \cdot \rangle$ stands for statistical average with respect to the distribution of the noise sources.

K is the probability density function of the uniform distribution $[-50, 50] \times [-10, 10] \times [185, 200]$. It can be approximated by 200 point noise sources randomly distributed in the domain $[-50, 50] \times [-10, 10] \times [185, 200]$.

The power spectral density of the noise sources is Gaussian $\hat{F}(\omega) = \exp(-\omega^2)$.

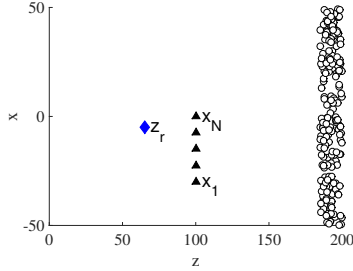


Figure 1: Geometric set-up: receivers (triangles), reflector (diamond), noise sources (circles).

Numerically, this can be simulated by sampling N points $(\mathbf{y}_s)_{s=1,\dots,N}$ independently according to the prescribed uniform distribution and each point emits independent and identically distributed signals $n_s(t)$ with stationary Gaussian statistics, mean zero, and covariance function $\langle n_s(t) n_s(t') \rangle = F(t - t')$:

$$n(t, \mathbf{x}) = \frac{1}{\sqrt{N}} \sum_{s=1}^N n_s(t) \delta(\mathbf{x} - \mathbf{y}_s) \quad (3)$$

There are five receivers: $\mathbf{x}_j = (-30 + 7.5(j - 1), 0, 100)$, $j = 1, \dots, 5$ (see Figure 1).

2) Preliminaries.

The homogeneous three-dimensional Green's function $\hat{G}_0(\omega, \mathbf{x}, \mathbf{y})$ is solution of

$$\Delta_{\mathbf{x}} \hat{G}_0 + \frac{\omega^2}{c_0^2} \hat{G}_0 = -\delta(\mathbf{x} - \mathbf{y}), \quad \mathbf{x} \in \mathbb{R}^2 \quad (4)$$

with the Sommerfeld radiation condition. It is given by

$$\hat{G}_0(\omega, \mathbf{x}, \mathbf{y}) = \frac{1}{4\pi|\mathbf{x} - \mathbf{y}|} \exp\left(i\frac{\omega}{c_0}|\mathbf{x} - \mathbf{y}|\right) \quad (5)$$

In order to generate the data in the presence of a point-like reflector at $\mathbf{z}_r$, we consider the Born approximation:

$$\hat{G}(\omega, \mathbf{x}, \mathbf{y}) = \hat{G}_0(\omega, \mathbf{x}, \mathbf{y}) + \sigma_r \omega^2 \hat{G}_0(\omega, \mathbf{x}, \mathbf{z}_r) \hat{G}_0(\omega, \mathbf{z}_r, \mathbf{y})$$

Take $c_0 = 1$, $\mathbf{z}_r = (x_r = -5, y_r = 0, z_r = 65)$, and $\sigma_r = 10^{-3}$.

3) Empirical cross correlation. The recorded signals are $(u(t, \mathbf{x}_j))_{j=1, \dots, 5, t \in [0, T]}$ with

$$\begin{aligned} u(t, \mathbf{x}_j) &= \frac{1}{2\pi} \iint \hat{G}(\omega, \mathbf{x}_j, \mathbf{y}) \hat{n}(\omega, \mathbf{y}) \exp(-i\omega t) d\mathbf{y} d\omega \\ &= \frac{1}{2\pi\sqrt{N}} \sum_{s=1}^N \int \hat{G}(\omega, \mathbf{x}_j, \mathbf{y}_s) \hat{n}_s(\omega) \exp(-i\omega t) d\omega \end{aligned}$$

The empirical cross correlation of the signals recorded at $\mathbf{x}_1$ and $\mathbf{x}_2$ for the recording time T is

$$C_{T,N}(\tau, \mathbf{x}_1, \mathbf{x}_2) = \frac{1}{T - |\tau|} \int_{\tau_-}^{T - \tau_+} u(t, \mathbf{x}_1) u(t + \tau, \mathbf{x}_2) dt. \quad (6)$$

where $\tau_+ = \max(\tau, 0)$ and $\tau_- = -\min(\tau, 0)$. Its expectation with respect to the distribution of the emitted signals is

$$C_N(\tau, \mathbf{x}_1, \mathbf{x}_2) = \frac{1}{2\pi N} \sum_{s=1}^N \int d\omega \hat{F}(\omega) \overline{\hat{G}(\omega, \mathbf{x}_1, \mathbf{y}_s)} \hat{G}(\omega, \mathbf{x}_2, \mathbf{y}_s) e^{-i\omega\tau}. \quad (7)$$

The expectation of the empirical cross correlation with respect to the distribution of the emitted signals and the source positions is

$$C^{(1)}(\tau, \mathbf{x}_1, \mathbf{x}_2) = \frac{1}{2\pi} \int d\omega \int d\mathbf{y} K(\mathbf{y}) \hat{F}(\omega) \overline{\hat{G}(\omega, \mathbf{x}_1, \mathbf{y})} \hat{G}(\omega, \mathbf{x}_2, \mathbf{y}) e^{-i\omega\tau}. \quad (8)$$

If $T \gg 1$, then the theory predicts that

$$C_{T,N}(\tau, \mathbf{x}_1, \mathbf{x}_2) \xrightarrow{T \rightarrow \infty} C_N(\tau, \mathbf{x}_1, \mathbf{x}_2) \quad (9)$$

If $N \gg 1$, then the theory predicts that

$$C_N(\tau, \mathbf{x}_1, \mathbf{x}_2) \xrightarrow{N \rightarrow \infty} C^{(1)}(\tau, \mathbf{x}_1, \mathbf{x}_2) \quad (10)$$

If the illumination were isotropic (which is not the case here), then we would have:

$$C^{(1)}(\tau, \mathbf{x}_1, \mathbf{x}_2) \approx C_{\text{asy}}(\tau, \mathbf{x}_1, \mathbf{x}_2), \quad (11)$$

$$\text{with } \frac{\partial}{\partial \tau} C_{\text{asy}}(\tau, \mathbf{x}_1, \mathbf{x}_2) = -[F * G(\tau, \mathbf{x}_1, \mathbf{x}_2) - F * G(-\tau, \mathbf{x}_1, \mathbf{x}_2)], \quad (12)$$

up to a multiplicative constant.

4) Questions.

1. We first consider the case where $T \rightarrow +\infty$.
 - a. Compute and plot $\tau \rightarrow C_N(\tau, \mathbf{x}_5, \mathbf{x}_1)$ for $\tau \in [-200, 200]$.
 - b. Compute and plot the KM image:

$$\mathcal{I}_N(\mathbf{y}^S) = \sum_{j,l=1}^5 C_N(|\mathbf{x}_j - \mathbf{y}^S| + |\mathbf{y}^S - \mathbf{x}_l|, \mathbf{x}_j, \mathbf{x}_l) \quad (13)$$

for $\mathbf{y}^S$ in the plane (x, z) in a square window with size 20×20 and centered on the reflector location $\mathbf{z}_r$.

- c. Study the resolution properties of the image.

2. We consider the case where $T < +\infty$.

- a. Compute and plot $\tau \rightarrow C_{N,T}(\tau, \mathbf{x}_5, \mathbf{x}_1)$ for $\tau \in [-150, 150]$ for different values of T ($T \geq 500$).

In order to increase the stability, we can consider M (non-overlapping) time windows of size T , which give independent realizations of the recorded signals $(u^{(m)}(t, \mathbf{x}_j))_{j=1,\dots,5, t \in [0, T]}$, $m = 1, \dots, M$. We can then compute M empirical cross correlations $C_{T,N}^{(m)}$, $m = 1, \dots, M$:

$$C_{T,N}^{(m)}(\tau, \mathbf{x}_j, \mathbf{x}_l) = \frac{1}{T - |\tau|} \int_{\tau_-}^{T - \tau_+} u^{(m)}(t, \mathbf{x}_j) u^{(m)}(t + \tau, \mathbf{x}_l) dt$$

and compute the average $C_{T,N,M}$ of the M empirical cross correlations:

$$C_{T,N,M}(\tau, \mathbf{x}_j, \mathbf{x}_l) = \frac{1}{M} \sum_{m=1}^M C_{T,N}^{(m)}(\tau, \mathbf{x}_j, \mathbf{x}_l)$$

- b. Compute and plot the CC-KM image:

$$\mathcal{I}_{T,N,M}(\mathbf{y}^S) = \sum_{j,l=1}^5 C_{T,N,M}(|\mathbf{x}_j - \mathbf{y}^S| + |\mathbf{y}^S - \mathbf{x}_l|, \mathbf{x}_j, \mathbf{x}_l) \quad (14)$$

- c. Study the stability properties of the image (with respect to T and M).

References:

- [1] J. Garnier and G. Papanicolaou, Passive Imaging with Ambient Noise, Cambridge University Press, Cambridge, 2016.